

Original Article

Third-Order Resolvent Dynamical Systems for Mixed Variational Inequalities: Convergence and Stability Analysis

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Abstract: This paper investigates a class of resolvent dynamical systems for developing inertial proximal methods to solve mixed variational inequalities. Third-order dynamics introduce additional inertia and damping effects, enabling the modeling of complex real-world systems while balancing stability and convergence speed. By exploiting the equivalence between the stationary points of the proposed dynamical system and the solutions of mixed variational inequalities, we show that the system trajectories converge to the unique solution of the problem. The convergence properties of the proposed inertial proximal methods are analyzed under mild conditions, assuming only monotonicity of the underlying operators. Furthermore, using a Lyapunov function framework, this study establishes the global asymptotic and exponential stability of the equilibrium points without requiring explicit solutions of the system. The proposed implicit and explicit discretization schemes provide a continuous-time perspective for designing efficient algorithms for solving mixed variational inequalities. To the best of our knowledge, this is the first work to apply third-order dynamical systems to mixed variational inequalities, offering a foundation for future extensions to stochastic, nonmonotone, and nonconvex equilibrium problems.

Keywords: Variational inequalities; Resolvent dynamics; Inertial methods; Stability.



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1. Introduction

Variational mathematical models form the foundation of many physical, biological, and financial systems (Douglas & Rachford, 1956; Giannessi et al., 2001; Kinderlehrer & Stampacchia, 2000; Korpelevich, 2000). These models arise in various forms, including optimization problems, equilibrium problems, variational inequalities, complementarity problems, and fixed-point problems (Blum & Oettli, 1994; Dupuis & Nagurney, 1993; Noor et al., 2017; Noor & Noor, 2025). Variational inequality (VI) provides a general mathematical framework that can be reformulated using dynamical systems to analyze the existence and stability of solutions. Specifically, a variational inequality can be associated with an ordinary differential equation (ODE) such that the stationary (equilibrium) points of the dynamical system coincide with the solutions of the corresponding variational inequality. In recent years, dynamical systems associated with variational inequalities have attracted considerable attention as powerful tools for analyzing complex dynamics and optimization processes. These approaches typically employ resolvent operators and

projection operators, such as the proximal point algorithm and gradient projection methods (Bin-Mohsin et al., 2017; Hai, 2022; Noor, 2002; Noor, 2004; Noor et al., 2017).

Unlike single-objective optimization problems, variational inequalities involve vector-valued functions and are equivalent to optimization problems only when the mapping corresponds to the gradient of an objective function. It is well known that solutions to variational inequalities exist when either the constraint set is bounded or the associated mapping is strongly monotone. Variational inequalities that include nonlinear terms are referred to as mixed variational inequalities (MVIs), also known as variational inequalities of the second kind (Bin-Mohsin et al., 2017). Mixed variational inequalities arise in a wide range of applications involving multiple interacting agents, such as game theory, mechanics, economics, and operations research. The convergence of dynamical systems to solutions of MVIs depends on the initial conditions, as such systems may exhibit sensitivity to these conditions. This convergence can be ensured by establishing the equivalence between variational inequalities and fixed-point problems through the use of Euclidean projection and resolvent operators.

A variety of projection-based methods have been developed to solve different classes of variational inequalities. These include classical projection methods and their variants, such as Wiener–Hopf equations (Noor, 2004), two-step extragradient methods (Korpelevich, 2000), and hyperplane projection methods (Bello-Cruz & Hazaimah, 2022), which rely on constructing separating hyperplanes followed by projection steps. When the objective function is nonsmooth or contains discontinuities often due to constraints proximal gradient methods are employed to compute subgradients effectively (Bello-Cruz & Hazaimah, 2022; Korpelevich, 2000). Among the most widely used approaches for solving MVIs is the proximal point algorithm. Since proximal operators generalize projection operators, projection methods can be viewed as a special case of proximal approaches. Noor et al. (2017) introduced proximal methods and projected dynamical systems for variational inequalities, while Noor (2002) extended first-order resolvent dynamical systems to MVIs. Building on this work, Bin-Mohsin et al. (2017) proposed second-order resolvent dynamical systems for mixed variational inequalities.

In this paper, we extend this line of research by introducing third-order dynamical systems for solving MVIs using resolvent operators. Third-order ODEs have practical relevance, particularly in modeling electrical circuits involving transistors (Goeleven, 2008). While first-order dynamical systems (e.g., gradient descent) are stable but often slow, and second-order systems (e.g., heavy-ball and Nesterov methods) provide acceleration through inertia but require careful damping, third-order systems offer an additional degree of control. This enables a more flexible balance between inertia, damping, and system response, potentially leading to improved convergence behavior.

Accordingly, we aim to design a continuous-time dynamical system whose solution converges to that of the corresponding MVI from any initial condition. Based on this framework, we also derive discrete-time algorithms tailored to the same problem. The main contributions of this work are as follows: (i) to develop a class of mixed variational inequalities using finite difference schemes through implicit and explicit discretization of the associated dynamical system expressed via resolvent operators; (ii) to establish convergence rates of the proposed methods; and (iii) to analyze the global stability of solutions for the third-order dynamical system. To the best of our knowledge, this is the first study to employ third-order differential equations in modeling MVIs via resolvent operators.

2. Preliminaries and Notations

Some optimization-related basics and significant foundations are presented in this section from monotone operators' theory, dynamical systems theory, convex and variational analysis, see (Bauschke & Combettes, 2017 and Kinderlehrer & Stampacchia 2000) for more details. Let \mathcal{H} be a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. Let $T: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be a set-valued map with its domain denoted $\text{dom}(T) := \{x \in \mathcal{H}; T(x) \neq \emptyset\}$. For any maximal monotone operator T the *resolvent* operator associated with T is the full domain single-valued operator in \mathcal{H} given by $J_T := (I+T)^{-1}: \mathcal{H} \rightarrow \text{dom}(T)$ where $I: \mathcal{H} \rightarrow \mathcal{H}$ denotes the identity operator. We are interested in designing dynamical systems models to derive discrete-time schemes for finding solutions to the mixed variational inequality which can be formulated as: find $x^* \in \mathcal{H}$, such that;

$$\langle T(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0, \quad \forall x \in \mathcal{H} \quad (1)$$

where $T: \text{dom } \varphi \rightarrow \mathcal{H}$ is an operator and $\varphi: \mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$ is a proper ($\text{dom } \varphi \neq \emptyset$), lower semicontinuous convex function. If C is a closed and convex set in \mathcal{H} and $\varphi(x) = I_C(x)$ is the indicator

function of C then the resolvent operator is the metric projection of \mathcal{H} onto C (i.e., $J_\varphi \equiv \Pi_C$), and problem (1) is reduced to the classical variational inequality: find $x^* \in \mathcal{H}$ such that;

$$\langle T(x^*), x - x^* \rangle \geq 0, \forall x \in C, \quad (2)$$

Problem (1) is equivalent to the generalized equation (aka monotone inclusion):

$$\text{Find } x^* \in \mathcal{H} \text{ such that } 0 \in T(x^*) + \partial\varphi(x^*), \quad (3)$$

where the subdifferential mapping $\partial\varphi: \mathcal{H} \rightrightarrows \mathcal{H}$, defined as $\partial\varphi(x) := \{u \in \mathcal{H}; \varphi(y) \geq \varphi(x) + \langle u, y - x \rangle, \forall y \in \mathcal{H}\}$ is a maximal monotone operator. The inclusion (3) may be extended to finding an element of the sum of two monotone operators and for this, the classical forward-backward method [6] is the most well-known splitting method for solving such problems. A particular case of (3) is when the operator T is the gradient of a smooth function f , i.e.,

$$0 \in \nabla f(x^*) + \partial\varphi(x^*),$$

The latter inclusion is precisely a convex nonsmooth optimization problem

$$\min_{x \in \mathcal{H}} f(x) + \varphi(x),$$

Moreover, if $T \equiv 0$, then (1) is exactly the above minimization problem with a convex nonsmooth objective function, i.e., $\min_{x \in \mathcal{H}} \varphi(x)$. If $C^* = \{x \in \mathcal{H}; \langle x, y \rangle \geq 0, \forall y \in \mathcal{H}\}$ is a polar (aka dual or conjugate) cone of a convex cone C then the inequality (2) is equivalent to finding $x \in C$ such that

$$T(x) \in C^* \text{ and } \langle T(x), x \rangle = 0,$$

which is called the generalized complementarity problem (Bauschke & Combettes, 2017; Kinderlehrer & Stampacchia 2000 and Noor & Noor, 2025) If the operator T in (2) is smooth, then the following well known result holds and can be viewed as a first order optimality condition for minimizing smooth functions:

Theorem 1. Let C be a nonempty, convex and closed subset of \mathcal{H} . Let T be a smooth convex function. Then $x \in C$ is the minimum of the smooth convex $T(x)$ if and only if, $x \in C$ satisfies

$$\langle T'(x), y - x \rangle \geq 0, \forall y \in C,$$

where T' is the Frechet derivative of T at $x \in C$.

This theorem shows that the variational inequalities are natural links and analogous to the minimization of the convex differentiable functional subject to certain constraint which has led to study a more general framework of variational inequalities applied to non-constrained and non-smooth optimization problems. In the following, we state some useful definitions and properties for several kinds of monotone maps followed by well-known facts on resolvent and projection operators, mixed variational inequalities and global stability at an exponential rate of equilibrium points of resolvent dynamical systems.

Definition 1. The operator $T: \mathcal{H} \rightarrow \mathcal{H}$, is said to be:

(i) Monotone, if

$$\langle T(x) - T(y), x - y \rangle \geq 0, \forall x, y \in \mathcal{H},$$

(ii) Strictly monotone if the above inequality is strict for all $x \neq y$ in \mathcal{H} .

(iii) Strongly monotone if there exists a modulus $\lambda > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq \lambda \|x - y\|^2, \forall x, y \in \mathcal{H},$$

Notice that the implication (iii) \implies (i) holds, whereas the converse need not be true generally, meaning that monotonicity is a weaker property than strongly monotonicity.

Definition 2. The operator $T: \mathcal{H} \rightarrow \mathcal{H}$ is called Lipschitz continuous or L-Lipschitz if there exists some nonnegative $L \geq 0$, such that

$$\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in \mathcal{H}.,$$

Proposition 2 ([2]). Let C be nonempty closed convex subset of \mathcal{H} , and Π_C be the orthogonal projection onto C . For all $x, y \in \mathcal{H}$ and all $z \in C$ the following hold:

$$\|\Pi_C(x) - \Pi_C(y)\|^2 \leq \|x - y\|^2 - \|(x - \Pi_C(x)) - (y - \Pi_C(y))\|^2, \quad (i)$$

$$\langle x - \Pi_C(x), z - \Pi_C(x) \rangle \leq 0, \quad (ii)$$

This proposition tailored for projection operators and variational inequalities. In the light of Proposition 2, we have the following result drawing similar connections between the resolvent operator and mixed variational inequalities (1) in which it plays a crucial role for deriving the convergence of the proposed implicit and explicit inertial proximal methods, which is known as the resolvent lemma.

Lemma 3. Let φ be a proper convex lower semicontinuous function. For all $x \in \mathcal{H}$ the following inequality hold:

$$\langle x - J_\varphi(x), y - J_\varphi(x) \rangle + \varrho\varphi(y) - \varrho\varphi(J_\varphi(x)) \leq 0, \forall y \in \mathcal{H},$$

where J_φ is the resolvent operator which belongs to the feasible set C .

By applying Lemma 3, one can introduce the fixed-point formulation of mixed variational inequalities as follows.

Proposition 4 (Kinderlehrer & Stampacchia, 2000). Let J_φ be the resolvent operator for the proper convex lower semicontinuous function φ and $T: \mathcal{H} \rightarrow \mathcal{H}$ is the underlying operator. Then $x \in \mathcal{H}$ is a solution to the mixed variational inequality (1), i.e.,

$$\langle Tx, y - x \rangle + \varphi(x) - \varphi(y) \leq 0,$$

if and only if $x = J_\varphi(x - \lambda T(x))$.

Definition 3. The dynamical system converges to the solution set \mathcal{C}^* of the mixed variational inequality (1) if the trajectory $\mathbf{x}(t)$ satisfies

$$dist(x(t), \mathcal{C}^*) := \inf_{y \in \mathcal{C}^*} \|x - y\| \rightarrow 0, \forall t \geq 0.,$$

Definition 4. The dynamical system is said to be globally exponentially stable if any trajectory $x(t)$ of the dynamical system satisfies

$$\|x(t) - x^*\| \leq \rho \|x(t_0) - x^*\| \exp(-\eta(t - t_0)), \forall t \geq t_0,$$

where $\rho, \eta > 0$ are constants and do not depend on the initial point.

If the dynamical system is stable at the equilibrium point x^* in the Lyapunov sense then the dynamical system is globally asymptotically stable at that point. It is noted that globally exponentially stable means the system must be globally stable and converge fast.

3. Main Results

In this section, we invoke the fixed-point formulation to introduce a new resolvent dynamical system of the third order associated with MVIs (1) and to investigate some accompanying suitable discretizations forms. These continuous-time dynamical systems and their discrete-time counterparts suggest some inertial proximal methods for solving MVIs. These inertial implicit and explicit methods are constructed using the central finite difference and forward/backward finite difference schemes and its variants. The third-order resolvent dynamical system takes the following form. Consider the problem of finding a trajectory $x(t) \in \mathcal{H}$ such that

$$\begin{cases} \alpha \ddot{x} + \beta \dot{x} + \gamma \dot{x} + x = J_\varphi(x - \lambda T(x)) \\ x(t_0) = x_0 \\ \dot{x}(t_0) = x_1 \\ \ddot{x}(t_0) = x_2 \end{cases}, \quad (4)$$

where $\alpha, \beta, \gamma > 0$ are constants and $x(t)$ is the state variable. The differential system (4) recovers several existing dynamics-type approaches and projection-based algorithms for solving mixed variational inequalities. Following are some particular cases of the general system (4). If $\alpha = 0, \gamma = 1$, then (4) is reduced to the second-order resolvent dynamical system introduced and studied by Mohsin et al. (2017) as

$$\beta \ddot{x} + \dot{x} + x = J_\varphi(x - \lambda T(x)), x(t_0) = x_0, \dot{x}(t_0) = x_1,$$

If $\alpha = 0 = \beta$, then (4) recovers the resolvent dynamical system which was analyzed by Noor (2002),

$$\frac{dx}{dt} = \gamma [J_\varphi(x - \lambda T(x)) - x], x(t_0) = x_0,$$

If $\alpha = 0 = \beta = \gamma$, then the system (4) is equivalent to the classical gradient projection for smooth constrained optimization problems and projection-like methods for solving variational inequalities.

3.1. Iterative methods

We start, as in most standard ways, with the discretization of the space derivatives. Taking suitable discretization of (4), and by using the central finite difference, backward difference and forward difference schemes, we propose explicit and implicit forms which enable us to obtain the discretized counterpart of (4) of order three as a resolvent equation:

$$\begin{aligned} \alpha \frac{x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}}{2h^3} + \beta \frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \gamma \frac{x_n - x_{n-1}}{h} + x_{n+2}, \\ = J_\varphi(x_n - \lambda T(x_{n+2})), \end{aligned} \quad (5)$$

where h is the step size for the iterative process. This discrete scheme (5) suggests a new implicit iterative method for solving mixed variational inequalities (1) by the third order central difference formula.

Algorithm 3.1. For any $x_0 \in \mathcal{H}$, and for any nonnegative integer $n \in \mathbb{Z}_+$, compute x_{n+2} by the iterative process

$$x_{n+2} = J_\varphi[x_n - \lambda T(x_{n+2})], \quad (6)$$

This algorithm is inertial proximal-type method for solving (1). Using Lemma 3, Algorithm 3.1 can be rewritten in the variational equivalent form:

Algorithm 3.2. For any $x_0 \in \mathcal{H}$, and for any nonnegative integer $n \in \mathbb{Z}_+$, compute x_{n+2} by the iterative process

$$\left\langle \lambda T(x_{n+2}) + \frac{\alpha x_{n+2} - 2(\alpha - \beta h)x_{n+1} - 2(2\beta h - \gamma h^2)x_{n+2} + (\alpha + \beta h - \gamma h^2)x_{n-1} - \alpha x_{n-2}}{2h^3}, \right. \quad (7)$$

$$\left. y - x_{n+2} \right\rangle + \varrho \varphi(y) - \varrho \varphi(x_{n+2}) \geq 0, \forall y \in \mathcal{H}, \quad (7)$$

For the sake of simplicity, take $\alpha = \beta = \gamma = 1$, and by using different discretization, Algorithm 3.1 reduces to the following iterative:

$$\begin{aligned} \frac{x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}}{2h^3} + \frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \frac{x_n - x_{n-1}}{h} + x_{n+2}, \\ = J_\varphi(x_n - \lambda T(x_n)) \end{aligned},$$

which yields to the following recurrence formula

$$x_{n+2} = \frac{\hat{h}}{1+\hat{h}} J_{\varphi} \left[\left(1 - \frac{1}{h} + \frac{2}{h^2} \right) x_n - \lambda T(x_n) - \frac{(2h-2)x_{n+1} + (2+2h-2h^2)x_{n-1} - x_{n-2}}{2h^3} \right], \quad (8)$$

where $\hat{h} = 2h^3$. This is called an inertial explicit proximal method for solving MVIs (1). In this manner, we can suggest several explicit and implicit recursive methods for approximating solutions of MVIs (1). Furthermore, we can obtain a different discretization by using the central finite difference and this time with forward difference scheme rather than backward scheme as in (5), which allows us to propose a new iterative approach

$$\begin{aligned} \alpha \frac{x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}}{2h^3} + \beta \frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \gamma \frac{x_{n+1} - x_n}{h} + x_{n+2}, \\ = J_{\varphi}(x_n - \lambda T(x_{n+1})), \end{aligned} \quad (9)$$

which can be, equivalently, derived as the following inertial implicit proximal method:

Algorithm 3.3. For any $x_0 \in \mathcal{H}$, and for any nonnegative integer $n \in \mathbb{Z}_+$, compute x_{n+1} by

$$x_{n+2} = J_{\varphi}[x_n - \lambda T(x_{n+1})], \quad (10)$$

We note that by applying suitable discretization's, one can establish and design a variety of inertial explicit and implicit proximal-type methods for solving variational inequalities of the second kind (1). Convergence analyses for Algorithm 3.1 and global stability for the third-order dynamical system (4) are derived in the remaining part of this work.

3.2. Convergence of a Discrete System

In this section, we derive the convergence of a solution to the implicit iterative scheme (6) and equivalent variational form (7) given by Algorithm (3.1). However, other implicit (9) and explicit (8) proposed methods have a very similar arguments and follow the same guidelines except that there are some minor differences which is due to the values of the scalars formatting of α, β, γ , and also due to the existing diverse discretization schemes.

Theorem 5. Let $x \in \mathcal{H}$ be the solution of the mixed variational inequality (1) and x_{n+2} be the approximate solution using the inertial proximal method in (7). If T is monotone, then

$$\begin{aligned} (\alpha - \beta h + \gamma h^2) \| -x_{n+2} \|^2 \leq \alpha \| x - 2x_{n+1} + 2x_{n-1} - x_{n-2} \|^2 \\ + \gamma h^2 \| x_n - x_{n-1} + x - x_{n+2} \|^2 - \gamma h^2 \| x_n - x_{n-1} \|^2. \end{aligned} \quad (11)$$

Proof. Let $x \in \mathcal{H}$ be a solution of the mixed variational inequality (1). Since T is monotone operator and for any $\lambda > 0$, we obtain

$$\langle \lambda T(y), y - x \rangle + \varrho\varphi(y) - \varrho\varphi(x) \geq 0, \text{ for all } y \in \mathcal{H}, \quad (12)$$

Take $y = x_{n+2}$ in (12) and $y = x$ in (7) then we have, respectively,

$$\langle \lambda T(x_{n+2}), x_{n+2} - x \rangle + \varrho\varphi(x_{n+2}) - \varrho\varphi(x) \geq 0, \quad (13)$$

and

$$\left\langle \lambda T(x_{n+2}) + \frac{\alpha x_{n+2} - 2(\alpha - \beta h)x_{n+1} - 2(2\beta h - \gamma h^2)x_n + 2(\alpha + \beta h - \gamma h^2)x_{n-1} - \alpha x_{n-2}}{2h^3}, \right. \quad (14)$$

$$\left. x - x_{n+2} \right\rangle + \varrho\varphi(x) - \varrho\varphi(x_{n+2}) \geq 0, \quad (14)$$

Combining (13) and (14) together, we have

$$\langle \alpha x_{n+2} - 2(\alpha - \beta h)x_{n+1} - 2(2\beta h - \gamma h^2)x_n + 2(\alpha + \beta h - \gamma h^2)x_{n-1} - \alpha x_{n-2}, x - x_{n+2} \rangle \geq 0$$

Manipulating and rewriting the latter inequality as

$$\begin{aligned}
 0 &\leq \langle \alpha x_{n+2} - 2(\alpha - \beta h)x_{n+1} - 2(2\beta h - \gamma h^2)x_n + 2(\alpha + \beta h - \gamma h^2)x_{n-1} - \alpha x_{n-2}, x - x_{n+2} \rangle \\
 &\leq \langle \alpha(x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}) + 2\beta h(x_{n+1} - 2x_n + x_{n-1}) \\
 &\leq \alpha \langle x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}, x - x_{n+2} \rangle + 2\beta h \langle x_{n+1} - 2x_n + x_{n-1}, x - x_{n+2} \rangle \\
 &\quad + 2\gamma h^2 \langle x_n - x_{n-1}, x - x_{n+2} \rangle.
 \end{aligned}$$

Invoking the properties and relationships between vector norms and vector inner products by using the norm of addition of vectors followed from the law of cosine i.e., $2\langle x, y \rangle = \|x + y\|^2 - \|x\|^2 - \|y\|^2$ and $2\langle x, y \rangle \leq \|x\|^2 + \|y\|^2$. Hence, the last line of the above inequality (15) can be written as

$$\begin{aligned}
 0 &\leq \alpha (\|x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2} + x - x_{n+2}\|^2 - \|x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 \\
 &\quad - \|x - x_{n+2}\|^2) + \beta h (\|x_{n+1} - 2x_n + x_{n-1}\|^2 + \|x - x_{n+2}\|^2) \\
 &\quad + \gamma h^2 (\|x_n - x_{n-1} + x - x_{n+2}\|^2 - \|x_n - x_{n-1}\|^2 - \|x - x_{n+2}\|^2) \\
 &= \alpha \|x - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 - \alpha \|x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 - \alpha \|x - x_{n+2}\|^2 \\
 &\quad + \beta h \|x_{n+1} - 2x_n + x_{n-1}\|^2 + \beta h \|x - x_{n+2}\|^2 + \gamma h^2 \|x_n - x_{n-1} + x - x_{n+2}\|^2 \\
 &\quad - \gamma h^2 \|x_n - x_{n-1}\|^2 - \gamma h^2 \|x - x_{n+2}\|^2
 \end{aligned}$$

which implies

$$\begin{aligned}
 (\alpha - \beta h + \gamma h^2) \|x - x_{n+2}\|^2 &\leq \alpha \|x - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 \\
 &\quad - \alpha \|x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 + \beta h \|x_{n+1} - 2x_n + x_{n-1}\|^2, \\
 &\quad + \gamma h^2 \|x_n - x_{n-1} + x - x_{n+2}\|^2 - \gamma h^2 \|x_n - x_{n-1}\|^2.
 \end{aligned} \tag{15}$$

Thus, we have proved the convergence result using the technique of Alvarez and Attouch (2001) for the solution $x \in \mathcal{H}$ of the mixed variational inequality (1).

Theorem 6. *Let $x \in \mathcal{H}$ be the solution of (1). Let x_{n+2} be the approximate solution of Algorithm 3.1, Suppose that the operator T is monotone, then the generated sequence from (7) converges to the solution x , i.e., $\lim_{n \rightarrow \infty} x_{n+2} = x$.*

Proof. Let $x \in \mathcal{H}$ be a solution of (1). The previous result in Theorem 5 showed that $\{\|x - x_n\|\}$ is nonincreasing sequence and consequently $\{x_n\}$ is bounded. It also follows from (11) that we have

$$\begin{aligned}
 \gamma h^2 \sum_{n=2}^{\infty} \|x_n - x_{n-1}\|^2 &\leq (-\alpha + \beta h - \gamma h^2) \sum_{n=2}^{\infty} \|x - x_{n+2}\|^2 \\
 &\quad + \alpha \sum_{n=2}^{\infty} \|x - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 - \alpha \sum_{n=2}^{\infty} \|x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 \\
 &\quad + \beta h \sum_{n=2}^{\infty} \|x_{n+1} - 2x_n + x_{n-1}\|^2 + \gamma h^2 \sum_{n=2}^{\infty} \|x_n - x_{n-1} + x - x_{n+2}\|^2
 \end{aligned}$$

and consequently, after algebraic manipulations, the inequality reduced to

$$\sum_{n=2}^{\infty} \|x_n - x_{n-1}\|^2 \leq \frac{\beta}{\gamma h} (\|x - x_4\|^2 + \|x_3 - 2x_2 + x_1\|^2 + \|x_4 - 2x_3 + x_2\|^2) + \|x_2 - x_1\|^2$$

which implies that:

$$\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\|^2 = 0, \tag{16}$$

Let x^* be an accumulation point of the successive approximations $\{x_n\}$, hence there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that it converges to $x^* \in \mathcal{H}$. Replace x_n by the subsequence x_{n_k} in (7) and consider the long-term asymptotic behaviour of the subsequence (i.e., when $n_k \rightarrow \infty$) and using (16), we have

$$\langle T(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0, \text{ for all } x \in \mathcal{H},$$

which implies that x^* solves the mixed variational inequality (1) and

$$\begin{aligned} \|x - x_{n+2}\|^2 &\leq \frac{\alpha}{\alpha - \beta h + \gamma h^2} \|x - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 \\ &\quad - \frac{\alpha}{\alpha - \beta h + \gamma h^2} \|x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 + \frac{\beta h}{\alpha - \beta h + \gamma h^2} \|x_{n+1} - 2x_n + x_{n-1}\|^2 \\ &\quad + \frac{\gamma h^2}{\alpha - \beta h + \gamma h^2} \|x_n - x_{n-1} + x - x_{n+2}\|^2 - \frac{\gamma h^2}{\alpha - \beta h + \gamma h^2} \|x_n - x_{n-1}\|^2. \end{aligned}$$

Consequently,

$$\begin{aligned} \|x - x_{n+2}\|^2 &\leq \frac{\alpha}{\alpha - \beta h + \gamma h^2} \|x - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 \\ &\quad + \frac{\beta h}{\alpha - \beta h + \gamma h^2} \|x_{n+1} - 2x_n + x_{n-1}\|^2 \\ &\quad + \frac{\gamma h^2}{\alpha - \beta h + \gamma h^2} \|x_n - x_{n-1} + x - x_{n+2}\|^2 \\ &\leq \|x - 2x_{n+1} + 2x_{n-1} - x_{n-2}\|^2 + \|x_n - x_{n-1} + x - x_{n+2}\|^2 \end{aligned} \quad (17)$$

Thus, it follows from the above inequality that the sequence $\{x_{n+2}\}$ has exactly one accumulation point x^* and $\lim_{n \rightarrow \infty} x_{n+2} = x^*$, the required result.

3.3. Stability of the dynamical system

A system is called stable if its long-term behavior (i.e., dynamics) tend to stay somewhere irrespective of the initial conditions. Next, we prove the global asymptotic stability using the Lyapunov stability theory for the proposed third-order differential equation (4). Choose a candidate Lyapunov function of the form:

$$V(x) = \frac{1}{2}(\dot{x}^2 + x^2),$$

Hence, the derivative of the Lyapunov function $\dot{V}(x) = \dot{x}\ddot{x} + x\dot{x}$. For all $x \neq 0$, we have

$$\begin{aligned} \dot{V}(x) &= \dot{x}\ddot{x} + x\dot{x} \\ &= \frac{\dot{x}}{\beta}(-\alpha\ddot{x} - \gamma\dot{x} - x + J(x - \lambda T(x))) + x\dot{x} \\ &= -\frac{\alpha}{\beta}(\dot{x}\ddot{x}) - \frac{\gamma}{\beta}(\dot{x}\dot{x}) - \left(1 - \frac{1}{\beta}\right)x\dot{x} + \frac{\dot{x}}{\beta}J(x - \lambda T(x)) \\ &< 0 \end{aligned}$$

It is clear to see that the Lyapunov function $V(x) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 \geq 0$ is non-negative, and $V(x) = 0$ if and only if $x = 0$, which implies that $V(x)$ is positive definite. Hence $\dot{V}(x) < 0$ for all $x \neq 0$ and $V(x)$ is positive definite, in the Lyapunov sense. Thus, the equilibrium point $x = 0$ is globally asymptotically stable for the resolvent dynamical system (4).

Theorem 7. Let T be a strongly monotone operator and Lipschitz continuous map, and K is a convex closed set. Then the dynamical system (4) has a unique equilibrium point that is globally exponentially stable, i.e., $\|x^ - x\|^2 \leq C \exp^{-(\mu - \epsilon)t}$, where $\mu > \epsilon$.*

Proof. Set $y = x^*$ in [Lemma 3] and $x = \alpha\ddot{x} + \beta\dot{x} + \gamma\dot{x} + x$ in (1) for all $\varrho > 0$, we have

$$\langle x - \lambda Tx - J, x^* - J \rangle + \varrho\varphi(x^*) - \varrho\varphi(J) \leq 0,$$

where $J = \alpha\ddot{x} + \beta\dot{x} + \gamma\dot{x} + x$, and

$$\lambda\langle Tx^*, x^* - J \rangle + \varrho\varphi(J) - \varrho\varphi(x^*) \leq 0,$$

Combine the last two inequalities together, we obtain

$$\langle x - \lambda Tx + \lambda Tx^* - (\alpha\ddot{x} + \beta\dot{x} + \gamma\dot{x} + x), x^* - (\alpha\ddot{x} + \beta\dot{x} + \gamma\dot{x} + x) \rangle \leq 0,$$

or,

$$\langle x - \lambda(Tx - Tx^*) - (\alpha\ddot{x} + \beta\dot{x} + \gamma\dot{x} + x), x^* - (\alpha\ddot{x} + \beta\dot{x} + \gamma\dot{x} + x) \rangle \leq 0,$$

Rearranging terms

$$\lambda\langle Tx^* - Tx, x^* - x \rangle - \lambda\langle Tx^* - Tx, \alpha\ddot{x} + \beta\dot{x} + \gamma\dot{x} \rangle - \langle \alpha\ddot{x} + \beta\dot{x} + \gamma\dot{x}, x^* - x \rangle + \|\alpha\ddot{x} + \beta\dot{x} + \gamma\dot{x}\|^2 \leq 0,$$

After using simple computational rearrangements and transformations, we have

$$\lambda\langle Tx^* - Tx, x^* - x \rangle - \alpha\langle \lambda(Tx^* - Tx) + x^* - x, \ddot{x} \rangle - \beta\langle \lambda(Tx^* - Tx) + x^* - x, \dot{x} \rangle + \beta\langle \dot{x}, \alpha\ddot{x} + \gamma\dot{x} \rangle + \gamma\langle \dot{x}, \alpha\ddot{x} + \beta\dot{x} \rangle \leq 0,$$

Since the terms $\alpha^2\|\ddot{x}\|^2, \beta^2\|\dot{x}\|^2, \gamma^2\|\dot{x}\|^2$ are all nonnegative, then the following inequality holds

$$\lambda\langle Tx^* - Tx, x^* - x \rangle - \alpha\langle \lambda(Tx^* - Tx) + x^* - x, \ddot{x} \rangle - \beta\langle \lambda(Tx^* - Tx) + x^* - x, \dot{x} \rangle - \gamma\langle \lambda(Tx^* - Tx) + x^* - x, \dot{x} \rangle + \alpha\langle \ddot{x}, \beta\dot{x} + \gamma\dot{x} \rangle + \beta\langle \dot{x}, \alpha\ddot{x} + \gamma\dot{x} \rangle + \gamma\langle \dot{x}, \alpha\ddot{x} + \beta\dot{x} \rangle \leq (17) \leq 0$$

Consequently,

$$\lambda\langle Tx^* - Tx, x^* - x \rangle - \alpha\langle \lambda(Tx^* - Tx) + x^* - x - (\beta\dot{x} + \gamma\dot{x}), \ddot{x} \rangle - \gamma\langle \lambda(Tx^* - Tx) + x^* - x - (\alpha\ddot{x} + \beta\dot{x}), \dot{x} \rangle \leq 0,$$

Rewriting the following relations $-\alpha\langle x^* - x, \ddot{x} \rangle = \alpha\frac{1}{2}\frac{d^3}{dt^3}\|x^* - x\|^2$, $-\beta\langle x^* - x, \dot{x} \rangle = \beta\frac{1}{2}\frac{d^2}{dt^2}\|x^* - x\|^2$, $-\gamma\langle x^* - x, \dot{x} \rangle = \gamma\frac{1}{2}\frac{d}{dt}\|x^* - x\|^2$, and using the assumption that T is Lipschitzian then the inequality (18) can be reduced to:

$$\lambda L\|x^* - x\|^2 - \alpha\langle \lambda(Tx^* - Tx) - (\beta\dot{x} + \gamma\dot{x}), \ddot{x} \rangle + \alpha\frac{1}{2}\frac{d^3}{dt^3}\|x^* - x\|^2 - \gamma\langle \lambda(Tx^* - Tx) - (\alpha\ddot{x} + \beta\dot{x}), \dot{x} \rangle + \gamma\frac{1}{2}\frac{d}{dt}\|x^* - x\|^2 \leq 0, \tag{18}$$

Multiply (19) by $\exp(\mu t)$, therefore

$$\lambda e^{\mu t} L\|x^* - x\|^2 - \alpha e^{\mu t} \langle \lambda(Tx^* - Tx) - (\beta\dot{x} + \gamma\dot{x}), \ddot{x} \rangle + \frac{\alpha}{2} \frac{d}{dt} \left(e^{\mu t} \frac{d^2}{dt^2} \|x^* - x\|^2 \right) - \gamma e^{\mu t} \langle \lambda(Tx^* - Tx) - (\alpha\ddot{x} + \beta\dot{x}), \dot{x} \rangle + \frac{\gamma}{2} \frac{d}{dt} (e^{\mu t} \|x^* - x\|^2) \leq 0, \tag{19}$$

Integrating (20) from t_0 to t , and moving some constant terms to the right-hand side, then the inequality (20) reduces to

$$\alpha \frac{d^2}{dt^2} \|x^* - x\|^2 + \beta \frac{d}{dt} \|x^* - x\|^2 + \gamma \|x^* - x\|^2 \leq 2C_1 e^{-\mu t}, \tag{20}$$

Where,

$$C_1 = e^{\mu t} (\lambda L\|x^* - x\|^2 - \alpha\langle \lambda(Tx^* - Tx) - (\beta\dot{x} + \gamma\dot{x}), \ddot{x} \rangle - \beta\langle \lambda(Tx^* - Tx) - (\alpha\ddot{x} + \gamma\dot{x}), \dot{x} \rangle - \gamma\langle \lambda(Tx^* - Tx) - (\alpha\ddot{x} + \beta\dot{x}), \dot{x} \rangle)$$

Similarly, multiply the inequality (21) by $[\exp(\mu - \epsilon)t]$, where $\mu > \epsilon > 0$, and integrating (21) twice from t_0 to t . Hence inequality (21) can be reduced, after suitable calculations, to

$$\|x^* - x\|^2 \leq (\text{Constant})e^{-(\mu-\epsilon)t},$$

Proving that the trajectory $x(t)$ converges to x^* with an exponential rate.

4. Conclusions

This study considers a new dynamical system approach via resolvent operators designed to approximate the solution to a given variational inequality of the second kind (i.e., mixed variational inequality). This is done, by exploiting the equivalence between the stationary points of the associated dynamical system and the solutions of the mixed variational inequality problem, i.e., by proving that trajectories of these dynamical systems converge to the unique solution of the mixed variational inequalities. It can be expected that the techniques described in this paper will be useful for more elaborate dynamical models, such as stochastic models, and that the connection between such dynamical models and the mixed variational inequalities will provide a deeper understanding of variational equilibrium problems since the proposed discrete-time algorithms can be considered as continuous-time perspectives for solving mixed variational inequalities. The stability analysis of the novel dynamical system technique has been investigated in the spirit of the Lyapunov function constructed in this framework. This approach usually, without the need to know the system's explicit solutions, provides qualitative behavior of the system around the equilibrium points. One of the advantages of this approach is studying changes over time for energy-like functions (Lyapunov functions) without solving the differential equation.

Despite their validity, combining third-order dynamics into mixed variational inequalities carries various challenges due to the computational complexity when proposing composite optimization algorithms for solving such systems. Future research directions may focus on developing efficient algorithms, integrating machine learning techniques for parameter estimation, and extending the framework to stochastic environments and/or to nonmonotone manners whether on operators or in line searches for linearly convergence of algorithms. The proposed implicit and explicit algorithms may be extended for a broader class of generalized equilibrium problems and even beyond the convexity scope to nonconvex equilibrium variational problems.

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